On the mathematical formulation of the motion of an inviscid compressible fluid in axial compressors II

By M. Z. v. Krzywoblocki

(Received July 1, 1958)

2.6. Differential Invariants

Above, it was assumed that the physical components of the absolute velocity $\vec{C}$ in the $\{r, \theta, \phi\}$-coordinate system are: C_r, C_u, C_m. Using expressions (2.1.24) and eqs. (2.3.8a, b) one can calculate the contravariant components of the vector $\vec{C}$:

$$A^1 = C_r\,;\ A^2 = R^{-1}C_u\,;\ A^3 = R_m^{-1}C_m\,. \tag{2.6.1}$$

Using eq. (2.1.26) jointly with the elements of the tensor g_{ij} (eqs. 2.3.8a, b) furnishes the covariant components of the vector $\vec{C}$:

$$A_1 = 1\cdot C_r + 0\cdot A^2 + \sin\phi C_m = C_r + \sin\phi C_m\,; \tag{2.6.2a}$$

$$A_2 = 0\cdot A^1 + RC_u + 0\cdot A^3 = RC_u\,; \tag{2.6.2b}$$

$$A_3 = R_m \sin\phi C_r + 0\cdot A^2 + R_m C_m = R_m(\sin\phi C_r + C_m)\,. \tag{2.6.2c}$$

Using the second equation (2.2.1) (since we are in (r, θ, ϕ) coordinate system) and the second eq. (2.2.1a) jointly with the components of the tensor g^{ij} (eq. 2.3.21) furnishes the contravariant components of the gradient referred to the system of $\vec{a}_i$ base vectors:

$$\vec{a}_1: (\cos^2\phi)^{-1}\frac{\partial}{\partial r} + 0\cdot\frac{\partial}{\partial\theta} - \sin\phi(R_m\cos^2\phi)^{-1}\frac{\partial}{\partial\phi}\,; \tag{2.6.3a}$$

$$\vec{a}_2: 0\cdot\frac{\partial}{\partial r} + R^{-2}\frac{\partial}{\partial\theta} + 0\cdot\frac{\partial}{\partial\phi}\,; \tag{2.6.3b}$$

$$\vec{a}_3: -\sin\phi(R_m\cos^2\phi)^{-1}\frac{\partial}{\partial r} + 0\cdot\frac{\partial}{\partial\theta} + (R_m^2\cos^2\phi)^{-1}\frac{\partial}{\partial\phi}\,. \tag{2.6.3c}$$

The contravariant components of the curl $\vec{C}$ are given by the eqs. (2.2.2a), (2.3.18) and (2.6.2a, b, c): referred to the system of base vectors, $\vec{a}_i$:

$$\vec{a}_1: (RR_m\cos\phi)^{-1}\left\{\frac{\partial}{\partial\theta}[R_m(\sin\phi\cdot C_r + C_m)] - \frac{\partial}{\partial\phi}(RC_u)\right\}\,; \tag{2.6.4a}$$

$$\vec{a}_2: (RR_m\cos\phi)^{-1}\left\{\frac{\partial}{\partial\phi}(C_r + \sin\phi\cdot C_m) - \frac{\partial}{\partial r}[R_m(\sin\phi\cdot C_r + C_m)]\right\}\,; \tag{2.6.4b}$$

$$\vec{a}_3: (RR_m\cos\phi)^{-1}\left[\frac{\partial}{\partial r}(RC_u) - \frac{\partial}{\partial\theta}(C_r + C_m\sin\phi)\right]\,. \tag{2.6.4c}$$

The divergence of the velocity vector $\vec{C}=\vec{C}\{C_r, C_u, C_m\}$ is given by the formula (2.2.5) in conjunction with eqs. (2.6.1):

$$\operatorname{div}\vec{C}=(RR_m\cos\phi)^{-1}\left\{\frac{\partial}{\partial r}(RR_m\cos\phi\cdot C_r)+\frac{\partial}{\partial\theta}(RR_m\cos\phi R^{-1}C_u)\right.$$
$$\left.+\frac{\partial}{\partial\phi}(RR_m\cos\phi R_m^{-1}C_m)\right\}, \tag{2.6.5}$$

or:

$$\operatorname{div}\vec{C}=(RR_m)^{-1}\left\{\frac{\partial}{\partial r}(RR_mC_r)+\frac{\partial}{\partial\theta}(R_mC_u)\right\}$$
$$+(RR_m\cos\phi)^{-1}\frac{\partial}{\partial\phi}(R\cos\phi\cdot C_m)\,. \tag{2.6.6}$$

It can be shown very easily that the above given formulas transform into the well-known formulas in cylindrical and spherical polar coordinates. In fact, from eqs. (2.6.3d, e, f), (2.6.4d, e, f) and (2.6.5) one derives the expressions:

(i) Cylindrical polar coordinates $\{R, \theta, z\}$ with $r=R$, $\phi=0$, $R_m\cos\phi d\phi\to=dz$ (eq. 2.3.22):
the components of the base vectors $\vec{a}_i$, referred to the $x^i=\{R, \theta, z\}$ coordinate system are:

$$\vec{a}_1:(1,0,0)\,;\ \vec{a}_2:(0,R,0)\,;\ \vec{a}_3:(0,0,1)\,; \tag{2.6.7a}$$

components of a gradient referred to the system of the base vectors $\vec{a}_i$ $(i=1,2,3)$ are:

$$\vec{a}_1:\frac{\partial}{\partial R}\ ;\ \vec{a}_2:R^{-2}\frac{\partial}{\partial\theta}\ ;\ \vec{a}_3:\frac{\partial}{\partial z}\ ; \tag{2.6.7b}$$

components of a gradient referred to the system of the unit base vectors $\{\vec{i}, \vec{j}, \vec{k}\}$ in the $x^i=\{R, \theta, z\}$-coordinate system are:

$$\vec{i}:\frac{\partial}{\partial R}\ ;\ \vec{j}:R^{-1}\frac{\partial}{\partial\theta}\ ;\ \vec{k}:\frac{\partial}{\partial z}\ ; \tag{2.6.7c}$$

components of curl $\vec{C}$ referred to the system of the base vectors $\vec{a}_i$ are:

$$\vec{a}_1:R^{-1}\frac{\partial}{\partial\theta}C_m-\frac{\partial}{\partial z}C_u\ ; \tag{2.6.8a}$$

$$\vec{a}_2:R^{-1}\left(\frac{\partial}{\partial z}C_r-\frac{\partial}{\partial R}C_m\right); \tag{2.6.8b}$$

$$\vec{a}_3:R^{-1}\left[\frac{\partial}{\partial R}(RC_u)-\frac{\partial}{\partial\theta}C_r\right]; \tag{2.6.8c}$$

components of curl $\vec{C}$ referred to the system of unit base vectors

$\{\vec{i}, \vec{j}, \vec{k}\}$:

$$\vec{i} : R^{-1}\frac{\partial}{\partial\theta}C_m - \frac{\partial}{\partial z}C_u \ ; \qquad (2.6.8d)$$

$$\vec{j} : \frac{\partial}{\partial z}C_r - \frac{\partial}{\partial R}C_m \ ; \qquad (2.6.8e)$$

$$\vec{k} : R^{-1}\left[\frac{\partial}{\partial R}(RC_u) - \frac{\partial}{\partial\theta}C_r\right] ; \qquad (2.6.8f)$$

divergence of $\vec{C}$:

$$\operatorname{div}\vec{C} = R^{-1}\left\{\frac{\partial}{\partial R}(RC_r) + \frac{\partial}{\partial\theta}C_u\right\} + \frac{\partial}{\partial z}C_m \ ; \qquad (2.6.9)$$

(ii) spherical polar coordinates $\{R, \theta, \phi\}$ with $R_m = R\sin\theta$, $r = R$, $\phi = 0$ (eq. 2.3.25): Components of a gradient referred to the $\{\vec{i}, \vec{j}, \vec{k}\}$-system:

$$\vec{i} : \frac{\partial}{\partial R} \ ; \quad \vec{j} : R^{-1}\frac{\partial}{\partial\theta} \ ; \quad \vec{k} : (R\sin\theta)^{-1}\frac{\partial}{\partial\phi} \ ; \qquad (2.6.10)$$

components of curl $\vec{C}$ referred to the $\{\vec{i}, \vec{j}, \vec{k}\}$ system:

$$\vec{i} : (R\sin\theta)^{-1}\left[\frac{\partial}{\partial\theta}(\sin\theta\cdot C_m) - \frac{\partial}{\partial\phi}C_u\right] ; \qquad (2.6.11a)$$

$$\vec{j} : (R\sin\theta)^{-1}\frac{\partial}{\partial\phi}C_r - R^{-1}\frac{\partial}{\partial R}(RC_m) \ ; \qquad (2.6.11b)$$

$$\vec{k} : R^{-1}\left[\frac{\partial}{\partial R}(RC_u) - \frac{\partial}{\partial\theta}C_r\right] ; \qquad (2.6.11c)$$

divergence of $\vec{C}$:

$$\operatorname{div}\vec{C} = R^{-2}\frac{\partial}{\partial R}(R^2C_r) + (R\sin\theta)^{-1}\cdot\frac{\partial}{\partial\theta}(C_u\sin\theta) + (R\sin\theta)^{-1}\frac{\partial}{\partial\phi}C_m \ . \qquad (2.6.12)$$

In the $\{\bar{r}, \theta, m\}$-coordinate system the physical components of the absolute velocity are: C_r, C_u, C_m. Using expressions (2.1.24) and eqs. (2.4.6a, b) one can calculate the contravariant components of the vector $\vec{C}$:

$$A^1 = C_r \ ; \quad A^2 = R^{-1}C_u \ ; \quad A^3 = C_m \ . \qquad (2.6.13)$$

Using eq. (2.1.26) jointly with the elements of the tensor g_{ij} (eq. 2.4.6a, b) furnishes the covariant components of the vector $\vec{C}$:

$$A_1 = C_r + \sin\phi\cdot C_m \ ; \qquad (2.6.14a)$$

$$A_2 = RC_u \ ; \qquad (2.6.14b)$$

$$A_3=\sin\phi\cdot C_r+C_m\ . \tag{2.6.14c}$$

Using the second equation (2.2.1) and the second eq. (2.2.1a) jointly with the components of the tensor g^{ij} (eq. 2.4.14) and of the vectors $\vec{a}_i$ (eq. 2.4.10), furnishes the contravariant components of a gradient:

$$\vec{a}_1:(\cos^2\phi)^{-1}\frac{\partial}{\partial \bar{r}}-\sin\phi(\cos^2\phi)^{-1}\frac{\partial}{\partial m}\ ; \tag{2.6.15a}$$

$$\vec{a}_2:R^{-2}\frac{\partial}{\partial\theta}\ ; \tag{2.6.15b}$$

$$\vec{a}_3:-\sin\phi(\cos^2\phi)^{-1}\frac{\partial}{\partial\bar{r}}+(\cos^2\phi)^{-1}\frac{\partial}{\partial m}\ ; \tag{2.6.15c}$$

contravariant components of curl $\vec{C}$ are:

$$\vec{a}_1:(R\cos\phi)^{-1}\left[\frac{\partial}{\partial\theta}(C_r\sin\phi+C_m)-\frac{\partial}{\partial m}(RC_u)\right]\ ; \tag{2.6.16a}$$

$$\vec{a}_2:(R\cos\phi)^{-1}\left[\frac{\partial}{\partial m}(C_r+\sin\phi C_m)-\frac{\partial}{\partial\bar{r}}(\sin\phi\cdot C_r+C_m)\right]\ ; \tag{2.6.16b}$$

$$\vec{a}_3:(R\cos\phi)^{-1}\left[\frac{\partial}{\partial\bar{r}}(RC_u)-\frac{\partial}{\partial\theta}(C_r+C_m\sin\phi)\right]\ ; \tag{2.6.16c}$$

divergence of $\vec{C}$:

$$\operatorname{div}\vec{C}=R^{-1}\left\{\frac{\partial}{\partial\bar{r}}(RC_r)+\frac{\partial}{\partial\theta}C_u\right\}+(\cos\phi)^{-1}\frac{\partial}{\partial m}(\cos\phi\cdot C_m)\ . \tag{2.6.17}$$

2.7. Vector Product

Using eqs. (2.2.12a, b, c) one can easily calculate the contravariant components, p^i, of the vector $\vec{p}=\vec{q}\times\vec{\omega}$, by putting $\vec{r}\equiv\vec{q}$, and $\vec{s}\equiv\vec{\omega}$, in various coordinate systems.

(i) $\{r,\theta,\phi\}$-coordinate system:

$$\begin{aligned}\vec{a}_1:p^1=(\cos^2\phi)^{-1}\Big\{&R^{-1}C_u\left[\frac{\partial}{\partial r}(RC_u)-\frac{\partial}{\partial\theta}(C_r+C_m\sin\phi)\right]\\&-R_m^{-1}C_m\left[\frac{\partial}{\partial\phi}(C_r+\sin\phi\cdot C_m)-\frac{\partial}{\partial r}[R_m(\sin\phi\cdot C_r+C_m)]\right]\Big\}\\&-\sin\phi(R_m\cos^2\phi)^{-1}\Big\{C_r\left[\frac{\partial}{\partial\phi}(C_r+\sin\phi\cdot C_m)-\frac{\partial}{\partial r}[R_m(\sin\phi\cdot C_r+C_m)]\right]\\&-R^{-1}C_u\left[\frac{\partial}{\partial\theta}[R_m(\text{sid}\,\phi\cdot C_r+C_m)]-\frac{\partial}{\partial\phi}(RC_u)\right]\Big\}\ ;\end{aligned} \tag{2.7.1}$$

$$\vec{a}_2 : p^2 = R^{-2}\left\{R_m^{-1}C_m\left[\frac{\partial}{\partial\theta}[R_m(\sin\phi\cdot C_r + C_m)] - \frac{\partial}{\partial\phi}(RC_u)\right]\right.$$

$$\left. -C_r\left[\frac{\partial}{\partial r}(RC_u) - \frac{\partial}{\partial\theta}(C_r - C_m\sin\phi)\right]\right\} ; \tag{2.7.2}$$

$$\vec{a}_3 : p^3 = -\sin\phi(R_m\cos^2\phi)^{-1}\left\{R^{-1}C_u\left[\frac{\partial}{\partial r}(RC_u) - \frac{\partial}{\partial\theta}(C_r + C_m\sin\phi)\right]\right.$$

$$\left. -R_m^{-1}C_m\left[\frac{\partial}{\partial\phi}(C_r + \sin\phi\cdot C_m) - \frac{\partial}{\partial r}[R_m(\sin\phi\cdot C_r + C_m)]\right]\right\}$$

$$+(R_m^2\cos^2\phi)^{-1}\left\{C_r\left[\frac{\partial}{\partial\phi}(C_r + \sin\phi\cdot C_m) - \frac{\partial}{\partial r}[R_m(\sin\phi\cdot C_r + C_m)]\right]\right.$$

$$\left. -R^{-1}C_u\left[\frac{\partial}{\partial\theta}[R_m(\sin\phi\cdot C_r + C_m)] - \frac{\partial}{\partial\phi}(RC_u)\right]\right\} ; \tag{2.7.3}$$

(ii) cylindrical polar coordinates (R, θ, z):

$$\vec{a}_1 : p^1 = R^{-1}C_u\left[\frac{\partial}{\partial R}(RC_u) - \frac{\partial}{\partial\theta}C_r\right] - C_m\left[\frac{\partial}{\partial z}C_r - \frac{\partial}{\partial R}C_m\right] ; \tag{2.7.4}$$

$$\vec{a}_2 : p^2 = R^{-2}\left\{C_m\left[\frac{\partial}{\partial\theta}C_m - \frac{\partial}{\partial z}(RC_u)\right] - C_r\left[\frac{\partial}{\partial R}(RC_u) - \frac{\partial}{\partial\theta}C_r\right]\right\} ; \tag{2.7.5}$$

$$\vec{a}_3 : p^3 = C_r\left[\frac{\partial}{\partial z}C_r - \frac{\partial}{\partial R}C_m\right] - R^{-1}C_u\left[\frac{\partial}{\partial\theta}C_m - \frac{\partial}{\partial z}(RC_u)\right] . \tag{2.7.6}$$

One may easily notice that the components p^i $(i=1, 2, 3)$ given in eqs. (2.7.4, 2.7.5, 2.7.6) are equivalent to the components of the vector product obtained in the usual way applied to an orthogonal system of coordinates with the use of the physical components of the velocity vector:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ C_r & C_u & C_m \\ \omega^1 & \omega^2 & \omega^3 \end{vmatrix} , \tag{2.7.7}$$

where the components ω^i $(i=1, 2, 3)$ are given by eqs. (2.6.8d, e, f), respectively.

One has only to keep in mind that eq. (2.7.5) must be multiplied by $|\vec{a}_2| = R$, in order to be in the full agreement with magnitude $\vec{j}(C_m\omega^1 - C_r\omega^3)$ obtained from expression (2.7.7);

(iii) $\{\bar{r}, \theta, m\}$-coordinate system:

$$\vec{a}_1 : p^1 = (\cos^2\phi)^{-1}\left\{R^{-1}C_u\left[\frac{\partial}{\partial\bar{r}}(RC_u) - \frac{\partial}{\partial\theta}(C_r + C_m\sin\phi)\right]\right.$$

$$\left. -C_m\left[\frac{\partial}{\partial m}(C_r + \sin\phi\cdot C_m) - \frac{\partial}{\partial\bar{r}}(\sin\phi\cdot C_r + C_m)\right]\right\}$$

$$-\sin\phi(\cos^2\phi)^{-1}\Big\{C_r\Big[\frac{\partial}{\partial m}(C_r+\sin\phi\cdot C_m)-\frac{\partial}{\partial \bar{r}}(\sin\phi\cdot C_r+C_m)\Big]$$

$$-R^{-1}C_u\Big[\frac{\partial}{\partial\theta}(C_r\sin\phi+C_m)-\frac{\partial}{\partial m}(RC_u)\Big]\Big\}\ ; \tag{2.7.8}$$

$$\vec{a}_2 : p^2=R^{-2}\Big\{C_m\Big[\frac{\partial}{\partial\theta}(C_r\sin\phi+C_m)-\frac{\partial}{\partial m}(RC_u)\Big]$$

$$-C_r\Big[\frac{\partial}{\partial\bar{r}}(RC_u)-\frac{\partial}{\partial\theta}(C_r+C_m\sin\phi)\Big]\Big\}\ ; \tag{2.7.9}$$

$$\vec{a}_3 : p^3=-\sin\phi(\cos^2\phi)^{-1}\Big\{R^{-1}C_u\Big[\frac{\partial}{\partial\bar{r}}(RC_u)-\frac{\partial}{\partial\theta}(C_r+C_m\sin\phi)\Big]$$

$$-C_m\Big[\frac{\partial}{\partial m}(C_r+\sin\phi\cdot C_m)-\frac{\partial}{\partial\bar{r}}(\sin\phi\cdot C_r+C_m)\Big]\Big\}$$

$$+(\cos^2\phi)^{-1}\Big\{C_r\Big[\frac{\partial}{\partial m}(C_r+\sin\phi\cdot C_m)-\frac{\partial}{\partial\bar{r}}(\sin\phi\cdot C_r+C_m)\Big]$$

$$-R^{-1}C_u\Big[\frac{\partial}{\partial\theta}(C_r\sin\phi+C_m)-\frac{\partial}{\partial m}(RC_u)\Big]\Big\}\ . \tag{2.7.10}$$

When treating the system of equations in $\{r, \theta, \phi\}$-system of coordinates, it was tacitly assumed that all the functions in the system are functions of (r, θ, ϕ). But it may happen that the functions in question are assumed to be functions of (R, θ, ϕ) with R given by eq. (2.3.3b). One can follow in this case the general formula; if f is a function of a finite number n_1 of variables $x, y, z, \cdots$, and if each of these variables is a function of a finite number n_2 of variables $u, v, w, \cdots$, (n_1 and n_2 being entirely independent), then (8):

$$f,_u|_{v,w,\cdots,}=f,_x|_{y,z,\cdots,}x,_u|_{v,w\ \cdots,}+f,_y|_{x,z,\cdots,}y,_u|_{v,w,\cdots,}+\cdots\ ; \tag{2.7.11}$$

$$f,_v|_{u,w,\cdots,}=f,_x|_{y,z,\cdots,}x,_v|_{u,w,\cdots,}+f,_y|_{x,z,\cdots,}y,_v|_{u,w,\cdots,}+\cdots\ , \tag{2.7.12}$$

where the dots at the end of each line indicate a finite number of terms, and

$$f=f(x(u, v, w, \cdots,), y(u, v, w, \cdots,), z(\cdots, \cdots), \cdots,)\ . \tag{2.7.13}$$

Let $f=f(R(r, \phi; R_m(r, \theta, \phi)), \theta, \phi)$ denote any function of (R, θ, ϕ), then one gets:

$$f,_r|_{\theta,\phi}=f,_R|_{\theta,\phi}\cdot(R,_r|_{\phi;R_m}+R,_{R_m}|_{r,\phi}\cdot R_{m,r}|_{\theta,\phi})\ ; \tag{2.7.14}$$

$$f,_\theta|_{r,\phi}=f,_R|_{\theta,\phi}\cdot R,_{R_m}|_{r,\phi}\cdot R_{m,\theta}|_{r,\phi}+f,_\theta|_{R,\phi}\ ; \tag{2.7.15}$$

$$f,_\phi|_{r,\theta}=f,_R|_{\theta,\phi}\cdot(R,_\phi|_{r,R_m}+R,_{R_m}|_{r,\phi}\cdot R_{m,\phi}|_{r,\theta})+f,_\phi|_{R,\theta}\ . \tag{2.7.16}$$

From eq. (2.3.3b) one gets:

$$R,_r|_{\phi;R_m}=1\ ;\quad R,_R\ |_{r,\phi}=(1-\cos\phi)\ ;\quad R,_\phi|_{r,R_m}=R_m\sin\phi\ , \tag{2.7.17}$$

which implies:

$$f,_r|_{\theta,\phi}=f,_R|_{\theta,\phi}\cdot[1+(1-\cos\phi)R_{m,r}|_{\theta,\phi}]\ ; \tag{2.7.18}$$

$$f,_\theta|_{r,\phi}=f,_R|_{\theta,\phi}(1-\cos\phi)\cdot R_{m,\theta}|_{r,\phi}+f,_\theta|_{R,\phi}\ ; \tag{2.7.19}$$

$$f,_\phi|_{r,\theta}=f,_R|_{\theta,\phi}\cdot(R_m\sin\phi+(1-\cos\phi)R_{m,\phi}|_{r,\theta})+f,_\phi|_{R,\theta}\ . \tag{2.7.20}$$

As another alternative consider that the function $R_m=R_m(R,\theta,\phi)$ is known, which immediately implies (see eq. 2.3.3b) that $R=R(r,\theta,\phi)$. Thus consider that $f=f(R(r,\theta,\phi),\theta,\phi)$, which furnishes:

$$f,_r|_{\theta,\phi}=f,_R|_{\theta,\phi}\cdot R,_r|_{\theta,\phi}\ ; \tag{2.7.21}$$

$$f,_\theta|_{r,\phi}=f,_R|_{\theta,\phi}\cdot R,_\theta|_{r,\phi}+f,_\theta|_{R,\phi}\ ; \tag{2.7.22}$$

$$f,_\theta|_{r,\theta}=f,_R|_{\theta,\phi}\cdot R,_\phi|_{r,\theta}+f,_\phi|_{R,\theta}\ . \tag{2.7.23}$$

The partial derivatives $R,_r$, $R,_\theta$, $R,_\phi$ must be calculated from eq. (2.3.3b) after the expression for $R_m=R_m(R,\theta,\phi)$ is substituted into eq. (2.3.3b).

3. MATHEMATICAL ASPECTS OF THE PROBLEM

3.1. Systems of Equations

As the representative system of equations we choose the system in the $\{r,\theta,\phi\}$-system of coordinates. After some elementary transformations the system of equations in question can be represented in the form: The Crocco equation (1.3.8) in conjunction with eq. (1.3.9) reduces to Euler's equation; it can be presented in the form of three scalar equations:

$$\begin{aligned}C_{r,t}+C_rC_{r,r}=&-(\rho\cos^2\phi)^{-1}(p,_r-R_m^{-1}\sin\phi\cdot p,_\phi)\\&-C_u(\cos^2\phi)^{-1}(C_{u,r}-R_m^{-1}\sin\phi\cdot C_{u,\phi})+C_u(\cos^2\phi\cdot R)^{-1}\cdot\\&\cdot[(RC_u),_r-R_m^{-1}\sin\phi\cdot(RC_u),_\phi]-R_m^{-1}C_m\cdot C_{r,\phi}-R^{-1}C_uC_{r,\theta}\\&+(R_m\cos^2\phi)^{-1}(C_m+C_r\sin\phi)^2\cdot R_{m,r}\\&-\sin\phi(\cos^2\phi\cdot R)^{-1}C_u(\sin\phi\cdot C_r+C_m)R_{m,\theta}-R_m^{-1}C_m^2(\cos\phi)^{-1}\ ;\end{aligned} \tag{3.1.1}$$

$$\begin{aligned}(R^{-1}C_u),_t+R^{-2}C_uC_{u,\theta}=&-R^{-2}\rho^{-1}p,_\theta-R^{-2}[C_r(RC_u),_r+C_mR_m^{-1}(RC_u),_\phi]\\&+R^{-2}R_m^{-1}C_m(C_r\sin\phi+C_m)R_{m,\theta}\ ;\end{aligned} \tag{3.1.2}$$

$$\begin{aligned}(C_mR_m^{-1}),_t+R_m^{-2}C_mC_{m,\phi}=&-(\rho R_m\cos^2\phi)^{-1}(-\sin\phi\cdot p,_r+R_m^{-1}p,_\phi)\\&-(R_m\cos^2\phi)^{-1}C_u(-C_{u,r}\sin\phi+R_m^{-1}C_{u,\phi})\\&+(R_mR\cos^2\phi)^{-1}C_u[-\sin\phi\cdot(RC_u),_r+R_m^{-1}(RC_u),_\phi]-R_m^{-1}C_rC_{m,r}\\&-R^{-1}C_uC_{m,\theta}-(R_m^2\cos^2\phi)^{-1}\cdot[(C_r+\sin\phi\cdot C_m)\cdot(C_r\sin\phi+C_m)R_{m,r}\\&+(R^{-1}C_u)(C_r\sin\phi+C_m)R_{m,\phi}]+C_m^2R_m^{-2}\sin\phi(\cos\phi)^{-1}\ ;\end{aligned} \tag{3.1.3}$$

continuity equation:

$$\rho,_t+\rho,_r C_r+(RR_m)^{-1}\{\rho(RR_m C_r),_r+(R_m\rho C_u),_\theta\}$$
$$+(R_m R\cos\phi)^{-1}(R\cos\phi\cdot\rho C_m),_\phi=0\ ; \qquad (3.1.4)$$

pressure-density relation:

$$p_0^{-1}p=(\rho_0^{-1}\rho)^\gamma \exp[c_v^{-1}(S-S_0)]\ . \qquad (3.1.5)$$

Provided that the entropy function $S=S(r, \theta, \phi; t)$ or $S=S(R, \theta, \phi; t)$ is a known function of the position and time, the system of 5 equations (3.1.1 to 3.1.5) in 5 functions C_r, C_u, C_m, ρ, p, is a determined system.

3.2. Axisymmetrical Steady Motion

Assume a steady flow having the axial symmetry, i.e., independent of the coordinate θ. This implies that all the derivatives with respect to θ vanish identically. The flow, being a three-dimensional, with the value of the velocity component C_u different from zero, and steady, possesses some kind of "helical" streamlines which coincide with free vortex lines (see Section 1.3) and along which the magnitudes H and $\vec{\omega}$ are constant. The distribution of the stagnation enthalpy H in the entrance cross section is assumed to be axially symmetric, i.e., depending only upon the radius $r=r_0\equiv R_0$ (with $\phi=\phi_0=0$) of the entrance circular cross section. A circular ring element of the entrance cross section, having a constant value of $H=H_0$ along its perimeter, originates—due to the motion of the fluid along the compressor—a surface of rotation on which the value of H is constant and equal to H_0. This may be called a "stream-surface". The streamlines originated in various points of the entrance circular ring of radius r_0, are situated on the corresponding "stream-surface". They are some kind of "helical curves". The body of rotation of the fluid inside the compressor is supposed to consist of such elementary stream-surfaces of revolution. A meridional cross-section through the axis of rotation of the body contains meridional cross sections of the stream surfaces.

The values of H and S are constant along each curve located on a stream-surface and consequently are constant along a meridional curve located on a stream-surface. The name "meridional curve" or "meridional cross-section" used in previous Sections, will be referred to a meridional curve located on a stream-surface. Consequently the values of H and S along such a curve are constant. From this standpoint, a meridional curve has properties analogous to a streamline. If in an isentropic flow field $\vec{\omega}\neq 0$, then from eq. (1.3.10) it follows that the stream-surfaces are "equi-vorticity surfaces". The vorticity vector is always directed normally to the streamline (and free vortex line) and has a constant value. Since the meridional curve is a locus of points of intersection of the streamlines with the meridional plane passing through the axis of rotation, to each point of the meridional curve there belongs a vorticity vector whose absolute value is constant but whose direction may vary from point to point.

The system of equations (1.3.8), (1.4.5) and (3.1.4) can be easily reduced to the system of equations in the steady, axially symmetrical flow. One can assume that all the dependent functions are functions of $\{R(r, \phi), \phi\}$.

3.3. Some Cases of Axially Un-symmetric Flow

Assume that in the initial cross-section the distributions of the velocity, pressure, and possibly density and temperature are non-uniform, i.e. they depend on the radius $r=r_0$ (or $R=R_0$) and the angle θ of the initial cross-section. A circle of a radius $r=r_1$ in the initial cross-section originates, due to the motion of the fluid along the compressor, a stream-surface. Assume that the flow is a stationary one, i.e., it does not change with time. Then the shape of the stream-surfaces in question does not change with time. But they are not necessarily the surfaces of rotation. The streamlines which are some kind of helices, situated on the stream surfaces, may have a variable pitch. Each of them may have different values of H and S, depending on the initial conditions. In practice it may happen that, due to viscosity and the action of the blades, the differences in H and S between streamlines located on one and the same stream-surface, may be annihilated beginning from some cross-section. Beginning from this cross-section, the stream-surface may be assumed to be a rotational one (axially symmetrical one). Meridional cross-sections of a stream-surface are, in general, not identical. To define a point on a meridional curve, one needs one coordinate more (usually the coordinate θ) than in the case of an axial symmetry. All the dependent functions, including H and S, are functions of $\{r, \theta, \phi\}$.

3.4. Initial Conditions

Two cases may be distinguished: all the functions are independent of time; all the functions depend on time.

(i) Independence of time.

Let the symbols $f^i(r, \theta, \phi)$ $(i=1, 2, \cdots, n)$ denote the functions in question $\{C_r, C_u, C_m, \rho, p\}$ and let ϕ^0 be a given number. Without a loss of generality one can refer the initial conditions to the initial cross-section of the compressor; assuming that in the vicinity of the initial cross-section there is a cylindrical shape of the duct, one can put down: $\phi^0=0$. Let $g^i(r, \theta)$ $(i=1, \cdots, n)$ be given functions of class A^1 in $T: \{r_1<r<r_2; \theta_1<\theta<\theta_2\}$. Then if

$$z^i = \varphi^i(r, \theta, \phi) \quad (i=1, \cdots, n), \tag{3.4.1}$$

is a solution of the system:

$$z_{j,\phi} = F^j(r, \theta, \phi; z_1, \cdots, z_n; z_{1,r}, \cdots, z_{n,\theta}) \quad (j=1, \cdots, n), \tag{3.4.2}$$

we shall require that the following condition be satisfied:

$$\varphi^i(r, \theta, \phi^0) \equiv g^i(r, \theta) \quad \text{for all } \{r, \theta\} \in T, \quad (i=1, \cdots, n). \tag{3.4.3}$$

(ii) Dependence on time.

Let the symbols $f^i(r, \theta, \phi; t)$ $(i=1, 2, \cdots, n)$ denote the functions in question $\{C_r, C_u, C_m, \rho, p\}$ and let t^0 be a given number. Without a loss of generality one can assume $t^0=0$. Let $g^i(r, \theta, \phi)$ be given functions of class A^1 in $T: \{r_1<r<r_2; \theta_1<\theta<\theta_2; \phi_1<\phi<\phi_2\}$. Then if

$$z_i = \varphi^i(r, \theta, \phi; t) \quad (i=1, \cdots, n), \tag{3.4.4}$$

is a solution of the system:

$$z_{j,t} = F^j(r, \theta, \phi; t; z_1, \cdots, z_n; z_{1,r}, \cdots, z_{n,\phi}) \quad (j=1, \cdots, n), \tag{3.4.5}$$

we shall require that the following conditions be satisfied:

$$\varphi^i(r, \theta, \phi; t^0) \equiv g^i(r, \theta, \phi) \quad \text{for all} \quad \{r, \theta, \phi\} \in T, \quad (i=1, \cdots, n). \tag{3.4.6}$$

3.5. Existence Theorems

In this section we shall cite various existence theorems referring to the systems of equations, given above. This will enable one to propose methods of solution of those systems, which may and should be related to the methods used in proving the existence theorems. This will give some assurance of the proper choice of the limiting process in question.

As usually, it is assumed that all the real functions, we treat, are decent, i.e., they are regular, analytic or of the class A^n $(n=0, 1, 2, \cdots, \infty)$, as required, etc.

Given the n-dimensional Euclidean space, E_n, of points P, where P is an n-tuple of real numbers $(x_1, x_2, \cdots, x_n)$. Below, one of the independent variables is singled out for special attention; it may be a time-coordinate, as opposed to several space coordinates, or it may be a variable for which an implicit relation is to be solved explicitly. For that reason, we shall take the number of independent variables in a general case as n+1, denoting the variables by $(x, y_1, \cdots, y_n)$.

In the presentation below, we follow closely Ref. (4).

In the space of two independent variables (x, y), and with a single unknown function z, the Problem of Cauchy may be stated as follows (4, p. 19):

Problem C: (a) Let $x_0(\mu)$, $y_0(\mu)$, $z_0(\mu)$ be functions of class A^1 in $M: \mu_1 < \mu < \mu_2$;

(b) Let $F(x, y, z, p, q)$ be a function of class A^0 (continuous) in a region U. Then it is required to establish the existence of a function $\phi(x, y)$ with the following properties:

(α) $\phi(x, y)$ is of class A^1 in a region R;

(β) for all $(x, y) \in R$, $(x, y, \phi(x, y), \phi_{,x}(x, y), \phi_{,y}(x, y)) \in U$ and

$$F(x, y, \phi(x, y), \phi_{,x}(x, y), \phi_{,y}(x, y)) \equiv 0 . \qquad (3.5.1)$$

That is, $z=\phi(x, y)$ is a solution in R of

$$F(x, y, z, z_{,x}, z_{,y})=0 . \qquad (3.5.2)$$

(γ) For all $\mu \in M_1 \leqq M$, $(x_0(\mu), y_0(\mu)) \in R$, and

$$\phi(x_0(\mu), y_0(\mu)) \equiv z_0(\mu) . \qquad (3.5.3)$$

That is, $z=\phi(x, y)$, considered as a surface, passes through the curve:

$$\Gamma : x=x_0(\mu); \quad y=y_0(\mu); \quad z_0(\mu) \ (\mu \in M_1) . \qquad (3.5.4)$$

Below, we shall give two other kinds of Cauchy problems.

Problem G (Initial value problem): Same as C except that (a) is replaced by (a_0) and (γ) by (γ_0):

(a_0) Let $g(y)$ be a function of class A^1 in $T: y_1 < y < y_2$ and x_0 be a given number;

(γ_0) For all $y \in T$, $(x_0, y) \in R$ and $g(y) \equiv \phi(x_0, y)$ in T.

Problem N: Same as G except that (b) is replaced by (b_0) and (β) by (β_0):

(b_0) $f(x, y, z, q)$ is of class A^0 in S;

(β_0) $z=\phi(x, y)$ is a solution in R of:

$$z_{,x}=f(x, y, z, z_{,y}) . \qquad (3.5.5)$$

Equation (3.5.5) is called the normal form of eq. (3.5.2).

The initial value problem for equations of normal type can be illustrated in the best way by means of the following theorem (4, p. 32):

Theorem 10.1: (a_∞) Let $g(y)$ be a function of class A^∞ for $|y-y_0|<\delta$ and let x_0 be a given number. Let $z_0=g(y_0)$ and $q_0=g'(y_0)$.

(b_∞) Let $f(x, y, z, q)$ be a function of class A^∞ in S_δ:

$$|x-x_0|<\delta; \quad |y-y_0|<\delta; \quad |q-q_0|<\delta .$$

Then there exists a unique function $\varphi(x, y)$ such that:

(α_∞) $\varphi(x, y)$ is of class A^∞ in some neighborhood R_{δ_1}:

$$|x-x_0|<\delta_1; \quad |y-y_0|<\delta_2 \quad \text{of} \quad (x_0, y_0) ;$$

(β_0) For all $(x, y) \in R_{\delta_1}$, $z=\varphi(x, y)$ is a solution of

$$z_{,x}=f(x, y, z, z_{,y}) ; \tag{3.5.6}$$

(γ_0) For all $y \in T_{\delta_1}$, $|y-y_0|<\delta_1$,

$$\varphi(x_0, y)\equiv g(y) . \tag{3.5.7}$$

The proof of this theorem is given in (4). To establish the convergence, the so-called method of majorants is used.

Theorem 10.2: Let (a_∞) and (b_∞) be given as in Theorem 10.1. Then there exists a uniformly convergent sequence of functions $\varphi^j(x, y)$ $(j=0, 1, 2, \cdots)$ of class A^∞ in a neighborhood R_{δ_1} of (x_0, y_0) such that:

$$\lim_{j\to\infty} \varphi^j(x, y)=\varphi(x, y) , \tag{3.5.8}$$

satisfied (α_∞), (β_0), (γ_0) and hence is the unique solution described in the previous theorem.

The proof of this theorem is in (9, p. 20). Germay does not use the method of majorants, but solves the characteristic equations of eq. (3.5.6) by the method of successive approximations.

The above theorems can be extended to $(n+1)$ variables (4, p. 38; 8, pp. 2–6). Consider now the system of "r" equations in "r" unknown functions and $(n+1)$ independent variables:

$$F_j(x, y_1, \cdots, y_n, z_1, \cdots, z_r, z_{1,x}, \cdots, z_{r,x}, z_{1,y_1}, \cdots, z_{r,y_n})=0$$
$$(j=1, \cdots, r) . \tag{3.5.9}$$

Solution of the system (3.5.9) is equivalent to the solution of the normal form of this system:

$$z_{j,x}=f^j(x, y_1, \cdots, y_n; z_1, \cdots, z_r; z_{1,y_1}, \cdots, z_{r,y_n}) \ (j=1, \cdots, r) . \tag{3.5.10}$$

The problem of Cauchy (4, p. 67) would be to determine a solution

$$z_j=\varphi_j(x, y_1, \cdots, y_n), \quad (j=1, \cdots, r) , \tag{3.5.11}$$

of the system (3.5.9), such that for a given set of functions

$$x=x^0(\mu_1, \cdots, \mu_n); \quad y_i=y_i^0(\mu_1, \cdots, \mu_n) ; \tag{3.5.12a}$$

$$z_j=z_j^0(\mu_1, \cdots, \mu_n) , \tag{3.5.12b}$$

with μ_k $(k=1, \cdots, n)$, being parameters, of class A^1 in a region M, where

$$\left\| \frac{\partial x^0}{\partial \mu_k} \frac{\partial y_1}{\partial \mu_k} \cdots \frac{\partial y_n}{\partial \mu_k} \right\| \tag{3.5.13}$$

is of rank n in M,

$$\varphi_j(x^0(\mu), y_1^0(\mu), \cdots, y_n^0(\mu))\equiv z_j(x^0(\mu), \cdots, y_n^0(\mu)) \text{ in } M . \tag{3.5.14}$$

We shall cite a theorem from (4, p. 67):

Theorem 18.1: (a_∞) Let $(x^0, y_1^0, \cdots, y_n^0)$ be a given point and $g^i(y_1, \cdots, y_n)$ $(i=1, 2, \cdots, r)$ be functions of class A^∞ in a neighborhood T_δ of this point;

(b_∞) Let

$$f^i(x, y_1, \cdots, y_n; z_1, \cdots, z_r; q_{11}, \cdots, q_{ik}, \cdots, q_{rn}) \quad (i=1, \cdots, r), \tag{3.5.15}$$

be functions of class A^∞ in a neighborhood S_δ of $(x^0, y_1, \cdots, y_n)$. Then there exists a unique set of functions $\varphi^i(x, y_1, \cdots, y_n)$ $(i=1, \cdots, r)$ such that:

(α_∞) $\varphi^i(x, y_1, \cdots, y_n)$ is of class A^∞ in a neighborhood R_δ of $(x^0, y_1^0, \cdots, y_n^0)$ $(i=1, \cdots, r)$;

(β_0) $z_i=\varphi^i(x, y_1, \cdots, y_n)$ is a solution of the system of eqs. (3.5.10), $(i=1, \cdots, r)$;

(γ_0) $\varphi^i(x^0, y_1, \cdots, y_n)=g^i(y_1, \cdots, y_n)$ for all $y \in T_\delta$, $(i=1, \cdots, r)$.

Proof of this theorem is given in Goursat (8, pp. 2–6). The method is exactly the same as that used for Theorem 10.1.

Theorem 18.2: Let (a_∞) and (b_∞) be given as above. Then there exist "r" uniformly convergent sequences of functions, $\varphi^{ij}(x, y_1, \cdots, y_n)$ $(i=1, \cdots, r;\ j=0, 1, 2, \cdots)$ of class A^∞ in a neighborhood of $(x_0, y_1^0, \cdots, y_n^0)$ such that

$$\lim_{j\to\infty} \varphi^{ij}(x, y_1, \cdots, y_n)=\varphi^i(x, y_1, \cdots, y_n) \quad (i=1, \cdots, r), \tag{3.5.16}$$

satisfy $(\alpha_\infty), (\beta_0), (\gamma_0)$ and hence are the unique functions of Theorem 10.1.

The proof of this theorem is given in (9, p. 31, ff), and the method of the proof is the same as that for Theorem 10.2.

The quasi-linear equation system, where $F_{ik}^j=F_{ik}^j(x, y_1, \cdots, y_n, z_1, \cdots, z_r)$:

$$\sum_{i=1}^{r} F_{i0}^j z_{i,x}+\sum_{k=1}^{n}\sum_{i=1}^{r} F_{ik}^j z_{i,yk}=F_0^j, \tag{3.5.17}$$

can be reduced to normal form:

$$z_{j,x}+\sum_{k=1}^{n}\sum_{i=1}^{r} f_{ik}^j z_{i,yk}=f_0^j, \tag{3.5.18}$$

provided either

$$|F_{i0}^j| \neq 0 \quad \text{or} \quad |F_{ik'}^j| \neq 0, \tag{3.5.19}$$

in a region U, or in some subregion of U for some k'.

In Ref. (4, p. 68a) we find a remark that Nagumo (5) established the existence of a solution of problem N for the system (3.5.2), where the functions f^i are of class A^0 in x and of class A^∞ in the other variables: he takes x real and the others complex. The system of equations given in Sections 3.1, 3.2 and 3.3 can be readily reduced to the normal form (3.5.5). Hence the Theorem 1, assures the existence of the local ("in the small") solution of that system, and the Nagumo proof of the problem N assures the existence of the solution "in the large" of that system under the assumptions in question. Some interesting existence theorems are in (6).

4. METHODS OF SOLUTION

In this section we shall briefly describe the methods of obtaining a solution of any of the systems, given in previous sections, following Ref. (13). At first, let us mention that a solution of problem N, discussed in section 3.5, leads to a solution of problem $C(N\rightarrow C)$ provided that the function F in problem C is of class A^1, that

$$F_{,p}\cdot[y_0'(\mu_0(y))]-F_{,q}\cdot[x_0'(\mu_0(y))]\neq 0 \text{ in } U\,,$$

and that $y_0'(\mu)\neq 0$ in M where $\mu_0(y)$ is the inverse of $y_0(\mu)$. For the proof, the reader is referred to Ref. (4, p. 21–25). Consequently, in the present Section we shall be concerned with the problem C. In this case the time "t" is supposed to be treated as another independent variable.

4.1. Quasi-linear Differential Equation in two Independent Variables

Consider a quasi-linear differential equation of the first order (13, p. 31):

$$a(x, y, f)f_{,x}+b(x, y, f)f_{,y}=h(x, y, f)\,. \tag{4.1.1}$$

Continuous differentiable solutions of this equation are represented by smooth surfaces $f=f(x, y)$. Eq. (4.1.1) defines in the $\{x, y, f\}$-space a vector field $\vec{p}=(a, b, h)$; we call $\vec{p}$ the Monge vector at the point (x, y, f). The vector $(f_{,x}; f_{,y}; -1)$ normal to the integral surface is perpendicular in each point to the Monge vector $\vec{p}=(a, b, h)$. Thus Monge vectors are also tangent vectors to the integral surfaces. All the planes tangent to the integral surfaces passing through a point $P(x, y, f)$ form a pencil of planes, the axis of whose is the Monge vector at P. A space directional field determined by the Monge vectors is given by the equations:

$$\frac{dx}{ds}=a(x, y, f)\,;\quad \frac{dy}{ds}=b(x, y, f)\,;\quad \frac{df}{ds}=h(x, y, f)\,. \tag{4.1.2}$$

The integral curves of eqs. (4.1.2) representable in terms of the parameter "s", $x=x(s)$, $y=y(s)$, $f=f(s)$, are called the characteristics "c" of the partial differential equation (4.1.1). Through each point

$\{x, y, f\}$ of the domain in question, there passes exactly one characteristic. The characteristics form a family of curves. The projections, " c' ", of the characteristics on the $\{x, y\}$-plane will be called the ground characteristics, or base characteristics.

The problem of integration of eq. (4.1.1) and the problem of integration of eq. (4.1.2) are equivalent in the following sense:

(a) each smooth surface spanned by a one-parameter family of characteristics is an integral surface of the differential equation (4.1.1);
(b) each integral surface of eq. (4.1.1) can be spanned by a one-parameter family of characteristics;
(c) when a characteristic has one point in common with an integral surface, it has all points in common, i.e., it belongs in the whole to the integral surface.

The following initial value problem will be treated: Given an initial space curve " k ": $x(t)$, $y(t)$, $f(t)$, or, which is equivalent, its projection, " k' ", on the $\{x, y\}$-plane: $x(t), y(t)$, together with the values of $f(t)$. The functions $x(t)$, $y(t)$, $f(t)$ suppose to possess the continuous first derivatives $\dot{x}, \dot{y}, \dot{f}$; moreover the condition: $\dot{x}^2(t)+\dot{y}^2(t)\neq 0$ should be preserved and the " k' " should not possess multiple points. One seeks in a certain neighborhood of " k " an integral surface $f=f(x, y)$, satisfying eq. (4.1.1), with continuous first derivatives of f, which surface passes through the given initial curve " k ".

In treating this problem one must be careful that the following conditions are preserved:

(a) the curve " k' " does not concide with any of the curves " c' ";
(b) the curve " k' " does not possess a point of tangency with any of the curves " c' ";
(c) the condition:

$$\dot{x}(t) : \dot{y}(t)\neq a : b \; ; \tag{4.1.3}$$

is preserved.

A solution of the eq. (4.1.1) will be found by transforming the differential equation (4.1.1) onto a difference equation. Let the subscripts " 1 " and " 2 " denote two points located on the initial curve " k' "; than the value of the function at the point " 3 " can be found from the equation:

$$a(f_3-f_1)(\Delta x)^{-1}+b(f_2-f_3)(\Delta y)^{-1}=h \; , \tag{4.1.4}$$

or

$$f_3=(h\Delta x\Delta y+af_1\Delta y-bf_2\Delta x)\cdot(a\Delta y-b\Delta x)^{-1} \; . \tag{4.1.5}$$

Concerning the proper choice of the grid, which question is a decisive one for the convergence of the process (4, p. V), the reader is referred to Ref. (13, pp. 16–26).

4.2. Quasi-Linear Differential Equation in more than two Independent Variables

Consider a quasi-linear differential equation:

$$a_1 f_{,x_1} + \cdots + a_n f_{,x_n} = h \ , \tag{4.2.1}$$

where the coefficients a_i $(i=1, \cdots, n)$ and h are the functions of $x_1, \cdots, x_n$, and f with continuous first partial derivatives. The above coefficients define in a $R_{n+1}(x_1, \cdots, x_n, f)$-space through the vectors $\vec{p}=(a_1, \cdots, a_n, h)$ a directional field called the Monge directional field. It is determined by means of the system of ordinary differential equations:

$$\frac{dx_1}{ds} = a_1, \cdots, \quad \frac{dx_n}{ds} = a_n, \quad \frac{df}{ds} = h \ . \tag{4.2.2}$$

The solutions of this system $x_1(s), \cdots, x_n(s), f(s)$ represent in the R_{n+1} space a n-parameter family of curves " c ", called the characteristics of the equation (4.2.1). Their projections, " c' ", on the $R_n(x_1, \cdots, x_n)$ are called the ground characteristics, or base characteristics.

Similarly like in Section 4.1, the problem of integration of eq. (4.2.1) and the problem of integration of the system (4.2.2) are equivalent in the following sense:

(a) each hypersurface in R_{n+1}-space spanned by a $(n-1)$-parametric family of characteristics is an integral surface $f=f(x_1, \cdots, x_n)$ of eq. (4.2.1);

(b) each integral surface of eq. (4.2.1) can be spanned by a $(n-1)$ parameter family of characteristics;

(c) when a characteristic has one point in common with an integral surface, it belongs in the whole to the integral surface.

The initial value problem can be formulated as follows:

A $(n-1)$-dimensional surface, " k_{n-1} ", is given in R_{n+1}-space by means of $x_i(t_1, \cdots, t_{n-1})$, $f(t_1, \cdots, t_{n-1})$, or, which is equivalent, its projection, " k'_{n-1} " in terms of x_i jointly with the values of $f(t)$. The functions x_i and f satisfy usual conditions concerning the derivatives, multiple points, etc.; the rank of the matrix $(\partial x_i/\partial t_k)$ should be equal to $(n-1)$. One seeks in the neighborhood of " k_{n-1} " an integral surface (n-dimensional hypersurface) $f=f(x_1, \cdots, x_n)$ with continuous first derivatives, which contains the surface " k_{n-1} ".

To have a solvable case, we assume that the following conditions are preserved:

(a) the space " k'_{n-1} " does not coincide with any of the ground manifolds " c'_{n-1} " and does not possess a point of tangency with the " c'_{n-1} " manifolds; one has:

$$D=\begin{vmatrix} a_1 \cdots\cdots\cdots\cdots\cdots a_n \\ \dfrac{\partial x_1}{\partial t_1} \cdots\cdots\cdots \dfrac{\partial x_n}{\partial t_1} \\ \text{--------------} \\ \dfrac{\partial x_1}{\partial t_{n-1}} \cdots\cdots\cdots \dfrac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} \neq 0\,; \qquad (4.2.3)$$

then the characteristics originated in the points of the given surface "k_{n-1}" span an integral surface, and this is the only integral surface which contains the surface "k_{n-1}".

Beginning from this point the procedure is an analogous one to that one described in Section 4.1.

4.3. Systems of Quasi-linear Equations of the First Order in two Independent Variables

Consider a system of two equations:

$$a_{11}u_{,x}+a_{12}v_{,x}+b_{11}u_{,y}+b_{12}v_{,y}=h_1\ , \qquad (4.3.1a)$$

$$a_{21}u_{,x}+a_{22}v_{,x}+b_{21}u_{,y}+b_{22}v_{,y}=h_2\ , \qquad (4.3.1b)$$

where $u=u(x, y)$, $v=v(x, y)$, and the coefficients a_{ik}, b_{ik}, h_i $(i, k=1, 2)$ are some defined functions of $\{x, y, u, v\}$. We assume that the usual differentiability conditions are satisfied. The determinant

$$\begin{vmatrix} a_{11}dy-b_{11}dx & a_{12}dy-b_{12}dx \\ a_{21}dy-b_{21}dx & a_{22}dy-b_{22}dx \end{vmatrix}=0\ , \qquad (4.3.2)$$

determines a directional field in $\{x, y\}$-domain in question. We assume that the equation resulting from the condition (4.3.2) furnishes in each point two real directions called the characteristic directions. In this case the corresponding system of equations (4.3.1a, b) is called a hyperbolic one, and the integral curves of the directional field determined by the condition (4.3.2) are called the characteristics of the system (4.3.1a, b).

The following initial value problem will be considered:

In the $\{x, y\}$-plane there is given a smooth curve "k" without any multiple point by means of a parametric representation $x(t)$, $y(t)$, jointly with the corresponding values of $u(t)$ and $v(t)$. The four functions x, y, u, v, are supposed to be continuous and differentiable, and $\dot{x}^2+\dot{y}^2\neq 0$. For the given initial values x, y, u, v, the system (4.3.1a, b) should be hyperbolic and in none of the points of the curve "k" the direction of the tangent should coincide with any of two characteristic directions. One seeks in a certain neighborhood of the curve "k" a continuous, differentiable solution $u(x, y)$, $v(x, y)$ of the system (4.3.1a, b) which on the curve "k" takes the prescribed values. This is called Problem I.

The condition (4.3.2) furnishes for the tangential directions of the characteristics λ=const. (1st family) and μ=const. (2nd family) the following two equations:

$$\left(\frac{dy}{dx}\right)_1=\sigma(x, y, u, v)\ ;\quad \left(\frac{dy}{dx}\right)_2=\rho(x, y, u, v)\ ;\quad \sigma\neq\rho\ . \quad (4.3.3)$$

This is equivalent to the system of the following two differential equations for

$$x(\lambda, \mu),\ y(\lambda, \mu):$$

$$x_{,\mu}\sigma-y_{,\mu}=0\ ,\quad x_{,\lambda}\rho-y_{,\lambda}=0\ . \qquad (4.3.4)$$

Moreover, the following two differential equations must be satisfied (13, p. 68):

$$\begin{vmatrix} \rho a_{11}-b_{11} & h_1x_{,\lambda}-a_{11}u_{,\lambda}-a_{12}v_{,\lambda} \\ \rho a_{21}-b_{21} & h_2x_{,\lambda}-a_{21}u_{,\lambda}-a_{22}v_{,\lambda} \end{vmatrix}=0\ ; \qquad (4.3.5)$$

$$\begin{vmatrix} \sigma a_{11}-b_{11} & h_1x_{,\mu}-a_{11}u_{,\mu}-a_{12}v_{,\mu} \\ \sigma a_{21}-b_{21} & h_2x_{,\mu}-a_{21}u_{,\mu}-a_{22}v_{,\mu} \end{vmatrix}=0\ . \qquad (4.3.6)$$

The system of equations, (4.3.4, 5, 6) called the characteristic system for the four functions $x(\lambda, \mu)$, $y(\lambda, \mu)$, $u(\lambda, \mu)$, $v(\lambda, \mu)$, can be represented in the following general form:

$$\alpha_{11}x_{,\lambda}+\alpha_{12}y_{,\lambda}+\alpha_{13}u_{,\lambda}+\alpha_{14}v_{,\lambda}=0\ ; \qquad (4.3.7a)$$

$$\alpha_{21}x_{,\lambda}+\alpha_{22}y_{,\lambda}+\alpha_{23}u_{,\lambda}+\alpha_{24}v_{,\lambda}=0\ ; \qquad (4.3.7b)$$

$$\alpha_{31}x_{,\mu}+\alpha_{32}y_{,\mu}+\alpha_{33}u_{,\mu}+\alpha_{34}v_{,\mu}=0\ ; \qquad (4.3.7c)$$

$$\alpha_{41}x_{,\mu}+\alpha_{42}y_{,\mu}+\alpha_{43}u_{,\mu}+\alpha_{44}v_{,\mu}=0\ . \qquad (4.3.7d)$$

We shall now define Problem II.

In the $\{\lambda, \mu\}$-plane there is given a curve "γ" jointly with the values of x, y, u, v, for the points of "γ". One seeks in a certain neighborhood of "γ" a solution of the system (4.3.7a, b, c, d) which takes on "γ" the prescribed values of x, y, u, v, and satisfies some conditions of continuity and differentiability (13, p. 70). Provided that the initial curves "k" and "γ" are associated one with each other, i.e., they correspond one to each other, the Problems I and II are equivalent in the following sense:

(a) each solution of the initial value problem I furnishes a solution of the initial value problem II;
(b) each solution of the initial value problem II furnishes a solution of the initial value problem I.

It is relatively easy to show that the following statement holds: The initial value problem I possess in the $\{x, y\}$-plane in the neighborhood of the curve "k", in particular in a curvilinear "rectangular" domain bounded by the characteristics $\lambda(x, y)=\text{const.}$, $\mu(x, y)=\text{const.}$, exactly one solution (13, p. 72).

The method, explained above, can be applied to the system of n-equations in two independent variables.

A numerical solution of the system (4.3.1a, b) may be obtained in more than one way; Sauer (13, pp. 73–86) discusses the following methods:

(i) solution of the system (4.3.7a, b, c, d) by a difference process and Picard iteration process following Friedrichs and Lewy; the proper choice of the grid is discussed;

(ii) reduction of the characteristic system (4.3.7a, b, c, d) to a system of second order partial differential equations and solution of the latter by means of the Picard iteration procedure following R. Courant;

(iii) graphico-numerical procedure of J. Massau (13, pp. 83–86), (14).

4.4. Systems of Quasi-linear Equations of the First Order in More than two Independent Variables

Consider a system of two quasi-linear differential equations in three independent variables:

$$a_{11}u_{,x}+a_{12}v_{,x}+b_{11}u_{,y}+b_{12}v_{,y}+c_{11}u_{,z}+c_{12}v_{,z}=h_1\ ; \tag{4.4.1}$$

$$a_{21}u_{,x}+a_{22}v_{,x}+b_{21}u_{,y}+b_{22}v_{,y}+c_{21}u_{,z}+c_{22}v_{,z}=h_2\ ; \tag{4.4.2}$$

where $u=u(x, y, z)$, $v=v(x, y, z)$ and the coefficients $a_{ik}, b_{ik}, c_{ik}, h_i$ $(i, k=1, 2)$ are the functions of x, y, z, u, v.

To the notion of the characteristic curves in the case of two independent variables there corresponds in the case of three or more independent variables the notion of the characteristic surfaces. For more details concerning these characteristic surfaces the reader is referred to Ref. (13, p. 149–153). They are given by the equation $z=k(x, y)$. Contrary to the results obtained in the case of two independent variables, in the present case the solutions of eqs. (4.4.1), (4.4.2) are in no way determined by the characteristic surfaces. The characteristic surfaces are discontinuity surfaces, similar to those known from the theory of the partial differential equations of the second order.

The method, described above, can be directly generalized to a system of p-differential equations ($p\geqq 2$) for p-functions $u_1, \cdots, u_p$, in n'' independent variables ($n\geqq 3$), $\{x_1, \cdots, x_n\}$. The characteristic surfaces are $(n-1)$-dimensional hypersurfaces $x_n=\kappa(x_1, \cdots, x_{n-1})$ in R_n-space and they result as the integral surfaces of a partial differential equation of the first order $R(\kappa_{,x_1}, \cdots, \kappa_{,x_{n-1}})=0$. The characteristic hypersurfaces are discontinuity surfaces of the solutions $u_i(x_1, \cdots, x_n)$. But the general theory pertinent to the solutions u_i depends in the present case upon the magnitude of n (even or odd).

5. CONCLUDING REMARKS

Above, the author presented a discussion on the subject of derivation of equations of motion in a curvilinear, non-orthogonal coordinate systems and of the possible numerical methods of solving these systems. The general scheme, in the case of a stationary flow condition, is the following:

Knowing the location and shape of the streamsurface, originated on a circle in the inlet cylindrical cross section of the compressor and all its characteristic properties (stagnation enthalpy, entropy, etc), one may find the meridional cross section of this surface, which curve possesses the same characteristic properties as the streamsurface. To define the system of equations in question, one has to know the coordinates of a point P, and the required parameter: radius of curvature of the meridional cross sectional curve. But, in order to know the shape of a streamsurface and its characteristic properties, one has to know the distribution pattern of the kinetic properties (velocity and density) and of the thermodynamic properties (pressure and temperature) of the fluid flow. As usually in the problems of such a nature, the complicated features of the nonlinearity appear here in the full light. The necessity of going in practice at the very beginning of the procedure into some kind of intuitive speculations and assumptions seems to be obvious.

The question becomes more complicated when there appears a non-uniform circumferential distribution of some of the characteristic properties of the fluid flow. Even in the stationary flow conditions, the streamlines located on the same streamsurface may have different characteristic properties. In order to have the proper distribution and variation of the characteristic properties along a meridional curve, one has to know the distribution and variation of the characteristic properties on the corresponding streamsurface. Here, there appears the problem of finding the pitch of the helical streamlines located on the streamsurface in question. This may be approximated by some kind of speculations on the ratio of the chord length of the rotor or stator to the circumference of the corresponding circle and number of revolution, etc.

The items mentioned above form only a small percentage of the difficulties appearing in the problem; their solution must be guessed before the full mathematical aspects of the problem can be brought up into the light.

University of Illinois

Bibliography

(1) Sokolnikoff, I. S.: Tensor Analysis. Theory And Applications. Chapman And Hall, Ltd., New York, 1951.

(2) Wills, A. P.: Vector Analysis With An Introduction To Tensor Analysis. Prentice-Hall, Inc., New York, 1931.

(3) Eisenhart, L. P.: An Introduction To Differential Geometry With The Use Of The Tensor Calculus. Princeton University Press, 1947.

(4) Bernstein, D. L.: Existence Theorems In Partial Differential Equations. Annals of Mathematics Study, No. 23, Princeton University Press, Princeton 1950.

(5) Nagumo, M.: Über Das Anfangswertproblem Partieller Differentialgleichungen. Jap. J. Math. 18, 41-47, 1942.

(6) Lichtenstein, L.: Über Einige Hilfssätze Der Potentialtheorie. I. Mathematische Zeitschrift, 23, 72-88, 1925.

Über Einige Existenzprobleme Der Hydrodynamik Homogener, Unzusammendrückbarer, Reibungsloser Flüssigkeiten Und Die Helmholtzschen Wirbelsätze. Ibid, 89-154, 1925.

Ergänzungen Zu Der Abhandlung: "Über Einige Existenzprobleme Der Hydrodynamik Homogener, Unzusammendrückbarer, Reibungsloser Flüssigkeiten Und Die Helmholtzschen Wirbelsätze." Ibid, 310-316, 1925.

Über Einige Existenzprobleme Der Hydrodynamik. Zweite Abhandlung. Nichthomogene, Unzusammendrückbare, Reibungslose Flüssigkeiten. Ibid, 26, 196-323, 1927.

Über Einige Existenzprobleme Der Hydrodynamik. Dritte Abhandlung. Permanente Bewegungen Einer Homogenen Inkompressiblen, Zähen Flüssigkeit. Ibid, 28, 387-415, 1928.

Über Einige Existenzprobleme Der Hydrodynamik. Vierte Abhandlung. Stetigkeitssätze. Einige Begründung Der Helmholtz-Kirchhoffschen Theorie Geradliniger Wirbelfäden. Ibid, 32, 608-640, 1930.

Grundlagen Der Hydromechanik. Berlin, Verlag Von J. Springer, 1929.

(7) Kamke, E.: Differentialgleichungen, Lösungsmethoden Und Lösungen. II. Partielle Differentialgleichungen Erster Ordnung Für Eine Gesuchte Funktion. Akademische Verlagsgesellschaft, Leipzig, 1944.

(8) Goursat, E.: Leçons Sur L'Intégration Des Équations Aux Dérivées Partielles Du Premier Ordre. Paris, Librairie Scientifique J. Hermann, 1921.

(9) Germay, R. H. J.: Intégration Par Approximation Successive Des Équations Aux Dérivées Partielles. Soc. Roy. Sc. Liege, Mem. (3), 12, No. 14, 1924.

(10) Kamke, E : Differentialgleichungen Reeller Funktionen. Akademische Verlagsgesellschaft, Leipzig, 1930.

(11) Perron, O.: Ein Neuer Beweis Des Fundamentalsatzes In Der Theorie Der Partiellen Differentialgleichungen Erster Ordnung. Mathematische Zeitschrift, Vol. 5, 1919, pp. 154-160.

(12) Levy, H. and Baggott,E. A.: Numerical Solutions Of Differential Equations. Dover Publications, Inc., 1950.

(13) Sauer, R.: Anfangswertprobleme Bei Partiellen Differentialgleichungen. Springer Verlag, Berlin-Göttingen-Heidelberg, 1952.

(14) Massau, J.: Mémoir Sur L'Intégration Graphique Des Équations Aux Dérivées Partielles. Gent, 1899. Also: Enzyklop. Der Math. Wiss. II, 3, 1, p. 162.